

Riemannian submersions from compact four manifolds

Xiaoyang Chen *

Abstract

We show that under certain conditions, a nontrivial Riemannian submersion from positively curved four manifolds does *not* exist. This gives a partial answer to a conjecture due to Fred Wilhelm. We also prove a rigidity theorem for Riemannian submersions with totally geodesic fibers from compact four-dimensional Einstein manifolds.

1 Introduction

A smooth map $\pi : (M, g) \rightarrow (N, h)$ is a Riemannian submersion if π_* is surjective and satisfies the following property:

$$g_p(v, w) = h_{\pi(p)}(\pi_*v, \pi_*w)$$

for any v, w that are tangent vectors in TM_p and perpendicular to the kernel of π_* .

A fundamental problem in Riemannian geometry is to study the interaction between curvature and topology. A lot of important work has been done in this direction. In this paper we study a similar problem for Riemannian submersions:

Problem: *Explore the structure of π under additional curvature assumptions of (M, g) .*

When (M, g) has constant sectional curvature, we have the following classification results ([8], [21], [22]).

Theorem 1.1. *Let $\pi : (M^m, g) \rightarrow (N, h)$ be a nontrivial Riemannian submersion (i.e. $0 < \dim N < \dim M$) with connected fibers, where (M^m, g) is compact and has constant sectional curvature c .*

1. *If $c < 0$, then there is no such Riemannian submersion.*
2. *If $c = 0$, then locally π is the projection of a metric product onto one of its factors.*
3. *If $c > 0$ and M^m is simply connected, then π is metrically congruent to the Hopf fibration, i.e., there exist isometries $f_1 : M^m \rightarrow \mathbb{S}^m$ and $f_2 : N \rightarrow \mathbb{P}(\mathbb{K})$ such that $pf_1 = f_2\pi$, where p is the standard projection from \mathbb{S}^m to projective spaces $\mathbb{P}(\mathbb{K})$.*

However, very little is known about the structure of π if (M, g) is not of constant curvature. In this paper we consider two different curvature conditions:

1. (M, g) has positive sectional curvature.

*The author is supported in part by NSF DMS-1209387.

2. (M, g) is an Einstein manifold.

When (M, g) has positive sectional curvature, we have the following important conjecture due to Fred Wilhelm (although never published anywhere).

Conjecture 1 *Let $\pi : (M, g) \rightarrow (N, h)$ be a nontrivial Riemannian submersion, where (M, g) is a compact Riemannian manifold with positive sectional curvature. Then $\dim(F) < \dim(N)$, where F is the fiber of π .*

By Frankel's theorem [7], it is not hard to see that Conjecture 1 is true if at least two fibers of π are totally geodesic. In fact, since any two fibers do not intersect with each other, Frankel's theorem implies that $2 \dim(F) < \dim(M)$. Hence $\dim(F) < \dim(N)$. If all fibers of π are totally geodesic, we have the following much stronger result:

Proposition 1.2. *Let $\pi : (M, g) \rightarrow (N, h)$ be a nontrivial Riemannian submersion such that all fibers of π are totally geodesic, where (M, g) is a compact Riemannian manifold with positive sectional curvature. Then $\dim(F) < \rho(\dim(N)) + 1$, where F is any fiber of π and $\rho(n)$ is the maximal number of linearly independent vector fields on S^{n-1} .*

Notice that we always have $\rho(\dim(N)) + 1 \leq \dim(N) - 1 + 1 = \dim(N)$ and equality holds if and only $\dim(N) = 2, 4$ or 8 .

Although not explicitly stated, Proposition 1.2 appears in [6]. For completeness, we will give a proof in section 3.

When $\dim(M) = 4$, Conjecture 1 is equivalent to the following conjecture.

Conjecture 2 *There is no nontrivial Riemannian submersion from any compact four manifold (M^4, g) with positive sectional curvature.*

In fact, suppose there exists such a Riemannian submersion $\pi : (M^4, g) \rightarrow (N, h)$. Then Conjecture 1 would imply $\dim(N) = 3$. Hence the Euler number of M^4 is zero. On the other hand, since (M^4, g) has positive sectional curvature, $H^1(M^4, \mathbb{R}) = 0$ by Bochner's vanishing theorem ([15], page 208). By Poincaré duality, the Euler number of M^4 is positive. Contradiction.

Let $\pi : (M, g) \rightarrow (N, h)$ be a Riemannian submersion. We say that a function f defined on M is basic if f is constant along each fiber. A vector field X on M is basic if it is horizontal and is π -related to a vector field on N . In other words, X is the horizontal lift of some vector field on N . Let H be the mean curvature vector field of the fibers and A be the O'Neill tensor of π . We denote by $\|A\|$ the norm of A , i.e., $\|A\|^2 = \sum_{i,j} \|A_{X_i} X_j\|^2$, where $\{X_i\}$ is a local orthonormal basis of the horizontal distribution of π . The next theorem gives a partial answer to Conjecture 2.

Theorem 1.3. *There is no nontrivial Riemannian submersion from any compact four manifold with positive sectional curvature such that either $\|A\|$ or H is basic.*

We emphasize that in Conjecture 1 the assumption that (M, g) has positive sectional curvature can *not* be replaced by (M, g) has positive sectional curvature *almost* everywhere, namely, (M, g) has nonnegative sectional curvature everywhere and has positive sectional curvature on an open and dense subset of M . Indeed, Let g be the metric on $S^2 \times S^3$ constructed by B. Wilking which has positive sectional curvature *almost* everywhere [23]. Then by a theorem

of K. Tapp [18], g can be extended to a nonnegatively curved metric \tilde{g} on $S^2 \times \mathbb{R}^4$ such that $(S^2 \times S^3, g)$ becomes the distance sphere of radius 1 about the soul. By Proposition 5.1, we get a Riemannian submersion $\pi : (S^2 \times S^3, g) \rightarrow (S^2, h)$, where h is the induced metric on the soul S^2 from \tilde{g} . This example shows that in Conjecture 1 the assumption that (M, g) has positive sectional curvature can *not* be replaced by (M, g) has positive sectional curvature *almost* everywhere.

Riemannian submersions are also important in the study of compact Einstein manifolds, for example, see [3]. Our next theorem gives a complete classification of Riemannian submersions with totally geodesic fibers from compact four-dimensional Einstein manifolds.

Theorem 1.4. *Suppose $\pi : (M^4, g) \rightarrow (N, h)$ is a Riemannian submersion, where (M^4, g) is a compact four-dimensional Einstein manifold. If all fibers of π are totally geodesic and have dimension 2, then locally π is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors, where $B^2(c)$ is a two-dimensional compact manifold with constant curvature c .*

If the dimension of the fibers of π is 1 or 3 (all fibers are not necessarily totally geodesic), then the Euler number of M^4 is zero. By a theorem of Berger [2, 13], (M^4, g) must be flat. Hence by a theorem of Walschap [21], locally π is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors.

Acknowledgment This paper is a part of my Ph.D thesis at University of Notre Dame [5]. The author would like to express gratitude to his advisor, Professor Karsten Grove, for many helpful discussions. He also thanks Professor Anton Petrunin for discussing the proof of Theorem 3.1. The author benefits a lot from his "Exercises in orthodox geometry" [16].

2 Preliminaries

In this section we recall some definitions and facts on Riemannian submersions which will be used in this paper. We refer to [14] for more details.

Let $\pi : (M, g) \rightarrow (N, h)$ be a Riemannian submersion. Then π induces an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$, where \mathcal{V} is tangent to the fibers and \mathcal{H} is the orthogonal complement of \mathcal{V} . We write $Z = Z^h + Z^v$ for the corresponding decomposition of $Z \in TM$. The O'Neill tensor A is given by

$$A_X Y = (\nabla_X Y)^v = \frac{1}{2}([X, Y])^v,$$

where $X, Y \in \mathcal{H}$ and are π -related to some vector field on N , respectively.

Fix $X \in \mathcal{H}$, define A_X^* by

$$\begin{aligned} A_X^* : \mathcal{V} &\rightarrow \mathcal{H} \\ V &\mapsto -(\nabla_X V)^h. \end{aligned}$$

Then A_X^* is the dual of A_X .

Define the mean curvature vector field H of π by

$$H = \sum_i (\nabla_{V_i} V_i)^h,$$

where $\{V_i\}_{i=1}^k$ is any orthonormal basis of \mathcal{V} and $k = \dim \mathcal{V}$.

Define the mean curvature form ω of π by

$$\omega(Z) = g(H, Z),$$

where $Z \in TM$. It is clear that $i_V \omega = \omega(V) = 0$ for any $V \in \mathcal{V}$.

We say that a function f defined on M is basic if f is constant along each fiber. A vector field X on M is basic if it is horizontal and is π -related to a vector field on N . In other words, X is the horizontal lift of some vector field on N . A differential form α on M is called to be basic if and only $i_V \alpha = 0$ and $\mathcal{L}_V \alpha = 0$ for any $V \in \mathcal{V}$, where $\mathcal{L}_V \alpha$ is the Lie derivative of α .

The set of basic forms of M , denoted by $\Omega_b(M)$, constitutes a subcomplex

$$d : \Omega_b^r(M) \rightarrow \Omega_b^{r+1}(M)$$

of the De Rham complex $\Omega(M)$. The basic cohomology of M , denoted by $H_b^*(M)$, is defined to be the cohomology of $(\Omega_b(M), d)$.

Proposition 2.1. *The inclusion map $i : \Omega_b(M) \rightarrow \Omega(M)$ induces an injective map*

$$H_b^1(M) \rightarrow H_{DR}^1(M).$$

Proof. See pages 33 – 34, Proposition 4.1 in [20]. □

3 Proof of Proposition 1.2 and Theorem 1.3

We first give a proof of Proposition 1.2.

Proof. Fix $p \in M$ and choose X_p to be any point in the unit sphere of \mathcal{H}_p . Extend X_p to be a unit basic vector field X . Since all fibers of π are totally geodesic, by O'Neill's formula ([14]), $K(X, V) = \|A_X^* V\|^2$ for any unit vertical vector field V . Since $K(X, V) > 0$ by assumption, we see that $A_X^* V \neq 0$ for any $V \neq 0$. Let v_1, v_2, \dots, v_k be any orthonormal basis of \mathcal{V}_p , where $k = \dim(F_p)$ and F_p is the fiber passing through p . Then $A_X^*(v_1), A_X^*(v_2), \dots, A_X^*(v_k)$ are linearly independent and are perpendicular to X_p . Since X_p could be any point in the unit sphere of \mathcal{H}_p , then we get k linearly independent vector fields on the unit sphere of \mathcal{H}_p . By the definition of $\rho(\dim N)$, we see that $\dim(F_p) = k \leq \rho(\dim(N)) < \rho(\dim(N)) + 1$. □

Remark 1. *It would be very interesting to know whether one can replace $\dim(F) < \dim(N)$ by $\dim(F) < \rho(\dim(N)) + 1$ in Conjecture 1. It would be the Riemannian analogue of Toponogov's Conjecture (page 1727 in [17]) and would imply that $\dim(N)$ must be even (In fact, if $\dim(N)$ is odd, then $\rho(\dim(N)) = 0$. Hence $\dim(F) < \rho(\dim(N)) + 1$ implies $\dim(F) = 0$ and hence π is trivial, contradiction). In particular, there would be no Riemannian submersion with one-dimensional fibers from even-dimensional manifolds with positive sectional curvature.*

Let (M^m, g) be an m -dimensional compact manifold with positive sectional curvature, $m \geq 4$ and (N^2, h) be a 2-dimensional compact Riemannian manifold. Now we are going to prove the following theorem which implies Theorem 1.3.

Theorem 3.1. *There is no Riemannian submersion $\pi : (M^m, g) \rightarrow (N^2, h)$ such that*

1. *the Euler numbers of the fibers are nonzero and*
2. *either $\|A\|$ or H is basic.*

Remark 2. *If Conjecture 1 is true, then there would be no Riemannian submersion $\pi : (M^m, g) \rightarrow (N^2, h)$, where (M^m, g) has positive sectional curvature and $m \geq 4$.*

Before we prove Theorem 3.1, we firstly show how to derive Theorem 1.3. The proof is by contradiction. Suppose there exists a nontrivial Riemannian submersion $\pi : (M^4, g) \rightarrow (N, h)$ such that either $\|A\|$ or H is basic, where (M^4, g) is a compact four manifold with positive sectional curvature. Since (M^4, g) has positive sectional curvature, $H^1(M^4, \mathbb{R}) = 0$ by Bochner's vanishing theorem ([15], page 208). By Poincaré duality, $\chi(M^4) = 2 + b_2(M^4)$ is positive. By a theorem of Hermann [12], π is a locally trivial fibration. Then $\chi(M^4) = \chi(N)\chi(F)$, where F is any fiber of π . It follows that $\dim(N) = 2$ and $\chi(F)$ is nonzero (hence all fibers have nonzero Euler numbers), which is a contradiction by Theorem 3.1.

The proof of Theorem 3.1 is again by contradiction. Suppose $\pi : (M^m, g) \rightarrow (N^2, h)$ be a Riemannian submersion satisfying the conditions in Theorem 3.1. By passing to its oriented double cover, we can assume that N^2 is oriented. The idea of the proof of Theorem 3.1 is to construct a nowhere vanishing vector field (or line field) on some fiber of π , which will imply the Euler numbers of the fibers are zero. Contradiction.

Since (M, g) has positive sectional curvature, by a theorem of Walschap [21], $\|A\|$ can not be identical to zero on M . Hence there exists $p \in M$ such that $\|A\|(p) \neq 0$.

If $\|A\|$ is basic, then $\|A\| \neq 0$ at any point on F_p , where F_p is the fiber at p . Let X, Y be any orthonormal oriented basic vector fields in some open neighborhood of F_p . Then $\|A_X Y\|^2 = \frac{1}{2}\|A\|^2 \neq 0$ at any point on F_p . Define a map s by

$$s : F_p \rightarrow TF_p$$

$$x \mapsto \frac{A_X Y}{\|A_X Y\|}(x).$$

Let Z, W be another orthonormal oriented basic vector fields. Then $Z = aX + bY$ and $W = cX + dY$, $ad - bc > 0$. Then

$$A_Z W = (ad - bc)A_X Y.$$

Hence s does not depend on the choice of X, Y . Then s is a nowhere vanishing vector field on F_p . Thus the Euler number of F_p is zero. Contradiction.

If H is basic, the construction of such nowhere vanishing vector field (or line field) is much more complicated. Under the assumption that H is basic, we firstly construct a metric \hat{g} on M^m such that $\pi : (M^m, \hat{g}) \rightarrow (N^2, h)$ is still a Riemannian submersion and all fibers are minimal submanifolds with respect to \hat{g} . Of course, in general \hat{g} can *not* have positive sectional curvature everywhere. However, the crucial point is that there exists some fiber F_0 such that \hat{g} has positive sectional curvature at all points on F_0 . Pick any fiber F_1 which is close enough to F_0 . Then using the classical variational argument, we construct a continuous codimension one distribution on F_1 . Thus the Euler number of F_1 is zero. Contradiction.

Now we are going to explain the proof of Theorem 3.1 in details. We firstly need the following lemmas:

Lemma 3.2. *Suppose ω is the mean curvature form of a Riemannian submersion from compact Riemannian manifolds. If ω is a basic form, then it is a closed form.*

Proof. See page 82 in [20] for a proof. \square

Lemma 3.3. *Suppose $\pi : (M^m, g) \rightarrow (N, h)$ is a Riemannian submersion such that H is basic, where (M^m, g) is a compact Riemannian manifold with positive sectional curvature. Then there exists a metric \hat{g} on M^m such that $\pi : (M^m, \hat{g}) \rightarrow (N, h)$ is still a Riemannian submersion and all fibers are minimal submanifolds with respect to \hat{g} . Furthermore, there exists some fiber F_0 such that \hat{g} has positive sectional curvature at all points on F_0 .*

Proof. The idea is to use partial conformal change of metrics along the fibers, see also page 82 in [20]. Let ω be the mean curvature form of π . Since H is basic, ω is a basic form. Then ω is closed by Lemma 3.2. So $[\omega]$ defines a cohomological class in $H_b^1(M^m)$. Because (M^m, g) has positive sectional curvature, $H_{DR}^1(M^m) = 0$ by Bochner's vanishing theorem ([15], page 208). By Proposition 2.1, we see that $H_b^1(M^m) = 0$. Then there exists a basic function f globally defined on M^m such that $\omega = df$. Define $\hat{f} = f - \max_{p \in M^m} f(p)$. Then $\max_{p \in M^m} \hat{f}(p) = 0$ and $\omega = d\hat{f}$. Let $\lambda = e^{\hat{f}}$ and define

$$\hat{g} = (\lambda^{\frac{2}{k}} g_v) \oplus g_h,$$

where $k = \dim(M^m) - \dim(N)$, g_v / g_h are the vertical / horizontal components of g , respectively.

Since the horizontal components of g remains unchanged, $\pi : (M^m, \hat{g}) \rightarrow (N, h)$ is still a Riemannian submersion. Now the mean curvature form $\hat{\omega}$ associated to \hat{g} is computed to be

$$\hat{\omega} = \omega - d \log \lambda = 0.$$

Hence all fibers of π are minimal submanifolds with respect to \hat{g} .

Let $\phi(p) = \lambda^{\frac{2}{k}}(p)$, $p \in M^m$. Then

$$\hat{g} = (\phi g_v) \oplus g_h.$$

Note for any $p \in M^m$, $0 < \phi(p) \leq 1$. Moreover, we have $\max_{p \in M^m} \phi(p) = 1$. Let $p_0 \in M^m$ such that $\phi(p_0) = 1$ and F_0 be the fiber of π passing through p_0 . Since f is a basic function on M^m , ϕ is also basic. Then $\phi \equiv 1$ on F_0 , which will play a crucial role in our argument below. Of course, in general \hat{g} can *not* have positive sectional curvature everywhere. However, by Lemma 3.4 below, we see that \hat{g} still has positive sectional curvature at all points on F_0 . (The reader should compare it to the following fact: Let $\hat{h} = e^{2f} h$ be a conformal change of h , where h is a Riemannian metric on M with positive sectional curvature. Then \hat{h} still has positive sectional curvature at those points where f attains its maximum value.)

Indeed, by Lemma 3.4 below, for any basic vector fields X, Y and vertical vector fields V, W , we have

$$\begin{aligned} \hat{K}(X + V, Y + W) \|(X + V) \wedge (Y + W)\|^2 &= \hat{R}(X + V, Y + W, Y + W, X + V) \\ &= R(X + V, Y + W, Y + W, X + V) + (\phi - 1)P(\nabla \phi, \phi, X, Y, V, W) \end{aligned}$$

$$\begin{aligned}
& +Q(\nabla\phi, \phi, X, Y, V, W) + [-g(W, W)g(\nabla_V \nabla\phi, X) \\
& +g(V, W)g(\nabla_W \nabla\phi, X) + g(V, W)g(\nabla_V \nabla\phi, Y) \\
& -g(V, V)g(\nabla_W \nabla\phi, Y)] + \frac{1}{2}[-Hess(\phi)(X, X)g(W, W) \\
& +2Hess(\phi)(X, Y)g(V, W) - Hess(\phi)(Y, Y)g(V, V)],
\end{aligned}$$

where $\hat{K}(X + V, Y + W)$ is the sectional curvature of the plane spanned by $X + V, Y + W$ with respect to \hat{g} and

$$\|(X + V) \wedge (Y + W)\|^2 = \hat{g}(X + V, X + V)\hat{g}(Y + W, Y + W) - [\hat{g}(X + V, Y + W)]^2.$$

Moreover, ∇ is the Levi-Civita connection and $Hess(\phi)$ is the Hessian of ϕ with respect to g . Also \hat{R}/R are the Riemannian curvature tensors with respect to \hat{g}/g , respectively. Furthermore, $P(\nabla\phi, \phi, X, Y, V, W)$, $Q(\nabla\phi, \phi, X, Y, V, W)$ are two functions depending on $\nabla\phi, \phi, X, Y, V, W$ and $Q(0, \phi, X, Y, V, W) \equiv 0$ (which will be very important for our purpose).

Since $\phi \equiv 1 = \max_{p \in M^m} \phi(p)$ on F_0 , we see that $\nabla\phi \equiv 0$ on F_0 . Hence $Q(\nabla\phi, \phi, X, Y, V, W) \equiv Q(0, \phi, X, Y, V, W) \equiv 0$ and $\nabla_V \nabla\phi \equiv 0, \nabla_W \nabla\phi \equiv 0$ on F_0 . Then at any point on F_0 , we have

$$\begin{aligned}
\hat{R}(X + V, Y + W, Y + W, X + V) &= R(X + V, Y + W, Y + W, X + V) \\
&+ \frac{1}{2}[-Hess(\phi)(X, X)g(W, W) + 2Hess(\phi)(X, Y)g(V, W) \\
&- Hess(\phi)(Y, Y)g(V, V)].
\end{aligned}$$

On the other hand, let

$$A = \begin{pmatrix} Hess(\phi)(X, X) & Hess(\phi)(X, Y) \\ Hess(\phi)(X, Y) & Hess(\phi)(Y, Y) \end{pmatrix}, B = \begin{pmatrix} g(W, W) & -g(V, W) \\ -g(V, W) & g(V, V) \end{pmatrix}.$$

Then

$$\hat{R}(X + V, Y + W, Y + W, X + V) = R(X + V, Y + W, Y + W, X + V) + \frac{1}{2}tr(-AB).$$

Since ϕ attains its maximum at any point on F_0 , we see that $-A$ is nonnegative definite on F_0 . It is easy to check that B is also nonnegative definite. Hence $tr(-AB) \geq 0$ (although $-AB$ is not nonnegative definite if $AB \neq BA$). Since g has positive sectional curvature everywhere on M^m by assumption, then at any point on F_0 , we see that

$$\hat{R}(X + V, Y + W, Y + W, X + V) \geq R(X + V, Y + W, Y + W, X + V) > 0.$$

Hence \hat{g} still has positive sectional curvature at all points on F_0 . \square

Lemma 3.4. *Let $\pi : (M^m, g) \rightarrow (N, h)$ be a Riemannian submersion and $g = g_v \oplus g_h$, where g_v / g_h are the vertical / horizontal components of g , respectively. Suppose ϕ is a positive basic function defined on M^m . Let $\hat{g} = (\phi g_v) \oplus g_h$. Suppose $\hat{\nabla} / \nabla$ are the Levi-Civita connections and \hat{R} / R are the Riemannian curvature tensors with respect to \hat{g} / g , respectively. Moreover, let $Hess(\phi)$ be the Hessian of ϕ with respect to g . Then for any horizontal vector fields X, Y (X, Y are not necessarily basic vector fields) and vertical vector fields V, W , we have*

$$\hat{\nabla}_X Y = \nabla_X Y.$$

$$\begin{aligned}
\hat{\nabla}_V W &= \nabla_V W - \frac{g(V, W)}{2} \nabla \phi + (\phi - 1)(\nabla_V W)^h. \\
\hat{\nabla}_V X &= \nabla_V X + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], V) \varepsilon_i. \\
\hat{\nabla}_X V &= \nabla_X V + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], V) \varepsilon_i,
\end{aligned}$$

where $\{\varepsilon_i\}_{i=1}^n$ is any orthonormal basis of the horizontal distribution with respect to g and $n = \dim(N)$.

Moreover, if X, Y are basic vector fields and V, W are vertical vector fields, then

$$\begin{aligned}
\hat{R}(X + V, Y + W, Y + W, X + V) &= R(X + V, Y + W, Y + W, X + V) \\
&+ (\phi - 1)P(\nabla \phi, \phi, X, Y, V, W) + Q(\nabla \phi, \phi, X, Y, V, W) \\
&+ [-g(W, W)g(\nabla_V \nabla \phi, X) + g(V, W)g(\nabla_W \nabla \phi, X) \\
&+ g(V, W)g(\nabla_V \nabla \phi, Y) - g(V, V)g(\nabla_W \nabla \phi, Y)] \\
&+ \frac{1}{2}[-Hess(\phi)(X, X)g(W, W) + 2Hess(\phi)(X, Y)g(V, W) \\
&- Hess(\phi)(Y, Y)g(V, V)],
\end{aligned}$$

where $P(\nabla \phi, \phi, X, Y, V, W), Q(\nabla \phi, \phi, X, Y, V, W)$ are two functions which depend on $\nabla \phi, \phi, X, Y, V, W$ and $Q(0, \phi, X, Y, V, W) \equiv 0$.

Proof. The proof is based on a lengthy computation and the following *Koszul's formula*:

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_X Y, Z) &= X(\hat{g}(Y, Z)) + Y(\hat{g}(Z, X)) - Z(\hat{g}(X, Y)) \\
&+ \hat{g}([X, Y], Z) - \hat{g}([Y, Z], X) - \hat{g}([X, Z], Y).
\end{aligned}$$

We just prove the fourth-fifth equalities in Lemma 3.4, others are left to the readers. In the computation below, we will use the following trick very often: If we encounter with anything like ϕX , we will rewrite $\phi X = X + (\phi - 1)X$. By rewriting it in this way, we can compare new curvature terms with odd terms. We will also use the fact that ϕ is a basic function very often.

Now let X, Y be horizontal vector fields (not necessarily basic) and $\{\varepsilon_i\}_{i=1}^n$ be any orthonormal basis of the horizontal distribution with respect to g . By *Koszul's formula*, we see that

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_X V, \varepsilon_i) &= X(\hat{g}(V, \varepsilon_i)) + V(\hat{g}(\varepsilon_i, X)) - \varepsilon_i(\hat{g}(X, V)) \\
&+ \hat{g}([X, V], \varepsilon_i) - \hat{g}([V, \varepsilon_i], X) - \hat{g}([X, \varepsilon_i], V) \\
&= V(\hat{g}(\varepsilon_i, X)) + \hat{g}([X, V], \varepsilon_i) - \hat{g}([V, \varepsilon_i], X) - \hat{g}([X, \varepsilon_i], V).
\end{aligned}$$

Since $\hat{g}_h = g_h$ and $\hat{g}_v = \phi g_v$, we get

$$\begin{aligned}
2g(\hat{\nabla}_X V, \varepsilon_i) &= Vg(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - \phi g([X, \varepsilon_i], V) \\
&= Vg(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - g([X, \varepsilon_i], V) + (1 - \phi)g([X, \varepsilon_i], V).
\end{aligned}$$

By *Koszul's formula* again, we see that

$$2g(\nabla_X V, \varepsilon_i) = Vg(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - g([X, \varepsilon_i], V).$$

Then

$$2g(\hat{\nabla}_X V, \varepsilon_i) = 2g(\nabla_X V, \varepsilon_i) + (1 - \phi)g([X, \varepsilon_i], V).$$

Hence

$$(\hat{\nabla}_X V)^h = (\nabla_X V)^h + \frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], V) \varepsilon_i.$$

Note that $\frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], V) \varepsilon_i$ does not depend on the choice of $\{\varepsilon_i\}_{i=1}^n$. By the similar argument above, we see that

$$(\hat{\nabla}_X V)^v = (\nabla_X V)^v + \frac{g(X, \nabla \phi)}{2\phi} V.$$

Hence

$$\hat{\nabla}_X V = \nabla_X V + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], V) \varepsilon_i.$$

The similar argument will also establish the first-third equalities in Lemma 3.4. We just mention that in the proof of these equalities, the fact that ϕ is a basic function and hence $V\phi = 0$ will be used very often.

Now we are going to prove the fifth equality in Lemma 3.4. In the following we always assume that X, Y are basic vector fields. First of all, we have

$$\begin{aligned} \hat{R}(X + V, Y + W, Y + W, X + V) &= \hat{R}(X, Y, Y, X) + \hat{R}(V, W, W, V) \\ &+ \hat{R}(X, W, W, X) + \hat{R}(Y, V, V, Y) + 2\hat{R}(X, Y, Y, V) + 2\hat{R}(Y, X, X, W) \\ &+ 2\hat{R}(X, Y, W, V) + 2\hat{R}(X, W, Y, V) + 2\hat{R}(V, W, W, X) + 2\hat{R}(W, V, V, Y). \end{aligned}$$

Since $\hat{g}_h = g_h$, $(M^m, \hat{g}) \rightarrow (N, h)$ is still a Riemannian submersion. Then by O'Neill's formula [14], we have

$$\begin{aligned} I_0 &= \hat{R}(X, Y, Y, X) = R_N(X, Y, Y, X) - \frac{3}{4} \hat{g}([X, Y]^v, [X, Y]^v) \\ &= R_N(X, Y, Y, X) - \frac{3}{4} g([X, Y]^v, [X, Y]^v) + \frac{3}{4} (1 - \phi) g([X, Y]^v, [X, Y]^v) \\ &= R(X, Y, Y, X) + \frac{3}{4} (1 - \phi) g([X, Y]^v, [X, Y]^v), \end{aligned}$$

where R_N is the Riemannian curvature tensor of (N, h) . On the other hand, by the first-fourth equalities in Lemma 3.4,

$$\begin{aligned} I_1 &= \hat{R}(V, W, W, V) = \hat{g}(\hat{\nabla}_V \hat{\nabla}_W W - \hat{\nabla}_W \hat{\nabla}_V W - \hat{\nabla}_{[V, W]} W, V) \\ &= \phi g(\hat{\nabla}_V [\nabla_W W - \frac{1}{2} g(W, W) \nabla \phi + (\phi - 1)(\nabla_W W)^h], V) \\ &\quad - \phi g(\hat{\nabla}_W [\nabla_V W - \frac{1}{2} g(V, W) \nabla \phi + (\phi - 1)(\nabla_V W)^h], V) \end{aligned}$$

$$\begin{aligned}
& -\phi g(\nabla_{[V,W]}W - \frac{1}{2}g([V,W],W)\nabla\phi + (\phi-1)(\nabla_{[V,W]}W)^h, V) \\
& = \phi g(\hat{\nabla}_V(\nabla_W W) - \hat{\nabla}_W(\nabla_V W) - \nabla_{[V,W]}W, V) \\
& \quad - \frac{1}{2}\phi[g(W,W)g(\hat{\nabla}_V\nabla\phi, V) - g(V,W)g(\hat{\nabla}_W\nabla\phi, V)] \\
& \quad + (\phi-1)\tilde{P}_1(\nabla\phi, \phi, X, Y, V, W) + \tilde{Q}_1(\nabla\phi, \phi, X, Y, V, W). \\
& = \phi g(\hat{\nabla}_V(\nabla_W W)^v + \hat{\nabla}_V(\nabla_W W)^h, V) - \phi g(\hat{\nabla}_W(\nabla_V W)^v + \hat{\nabla}_W(\nabla_V W)^h, V) \\
& \quad - \phi g(\nabla_{[V,W]}W, V) - \frac{1}{2}\phi[g(W,W)g(\hat{\nabla}_V\nabla\phi, V) - g(V,W)g(\hat{\nabla}_W\nabla\phi, V)] \\
& \quad + (\phi-1)\tilde{P}_1(\nabla\phi, \phi, X, Y, V, W) + \tilde{Q}_1(\nabla\phi, \phi, X, Y, V, W).
\end{aligned}$$

Since ϕ is a basic function, $g(\nabla\phi, V) = V\phi = 0$. Hence $\nabla\phi$ is a horizontal vector field. Then by the first-fourth equalities in Lemma 3.4, we see that

$$g(\hat{\nabla}_W\nabla\phi, V) = -g(\nabla_W V, \nabla\phi) + \frac{g(\nabla\phi, \nabla\phi)}{2\phi}g(W, V),$$

and

$$\begin{aligned}
I_1 &= \hat{R}(V, W, W, V) = \phi R(V, W, W, V) \\
&+ (\phi-1)\check{P}_1(\nabla\phi, \phi, X, Y, V, W) + \check{Q}_1(\nabla\phi, \phi, X, Y, V, W) \\
&= R(V, W, W, V) + (\phi-1)R(V, W, W, V) \\
&+ (\phi-1)\check{P}_1(\nabla\phi, \phi, X, Y, V, W) + \check{Q}_1(\nabla\phi, \phi, X, Y, V, W) \\
&= R(V, W, W, V) + (\phi-1)P_1(\nabla\phi, \phi, X, Y, V, W) + Q_1(\nabla\phi, \phi, X, Y, V, W),
\end{aligned}$$

where $P_1(\nabla\phi, \phi, X, Y, V, W)$, $Q_1(\nabla\phi, \phi, X, Y, V, W)$ are two functions depending on $\nabla\phi, \phi, X, Y, V, W$ and $Q_1(0, \phi, X, Y, V, W) \equiv 0$.

Since X is a basic vector field, $[X, W]$ is vertical. Hence by the first-fourth equalities in Lemma 3.4,

$$\begin{aligned}
I_2 &= \hat{R}(X, W, W, X) = \hat{g}(\hat{\nabla}_X\hat{\nabla}_W W - \hat{\nabla}_W\hat{\nabla}_X W - \hat{\nabla}_{[X,W]}W, X) \\
&= g(\hat{\nabla}_X[\nabla_W W - \frac{1}{2}g(W, W)\nabla\phi + (\phi-1)(\nabla_W W)^h], X) \\
&\quad - g(\hat{\nabla}_W[\nabla_X W + \frac{g(X, \nabla\phi)}{2\phi}W + \frac{1-\phi}{2}\sum_{i=1}^n g([X, \varepsilon_i], W)\varepsilon_i], X) \\
&\quad - g(\nabla_{[X,W]}W - \frac{1}{2}g([X, W], W)\nabla\phi + (\phi-1)(\nabla_{[X,W]}W)^h, X) \\
&= R(X, W, W, X) + (\phi-1)P_2(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_2(\nabla\phi, \phi, X, Y, V, W) - \frac{1}{2}Hess(\phi)(X, X)g(W, W),
\end{aligned}$$

where $P_2(\nabla\phi, \phi, X, Y, V, W)$, $Q_2(\nabla\phi, \phi, X, Y, V, W)$ are two functions depending on $\nabla\phi, \phi, X, Y, V, W$ and $Q_2(0, \phi, X, Y, V, W) \equiv 0$.

By the similar argument, we see that

$$\begin{aligned}
I_3 &= \hat{R}(Y, V, V, Y) = R(Y, V, V, Y) + (\phi - 1)P_3(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_3(\nabla\phi, \phi, X, Y, V, W) - \frac{1}{2}Hess(\phi)(Y, Y)g(V, V). \\
I_4 &= \hat{R}(X, Y, Y, V) = R(X, Y, Y, V) + (\phi - 1)P_4(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_4(\nabla\phi, \phi, X, Y, V, W). \\
I_5 &= \hat{R}(Y, X, X, W) = R(Y, X, X, W) + (\phi - 1)P_5(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_5(\nabla\phi, \phi, X, Y, V, W). \\
I_6 &= \hat{R}(X, Y, W, V) = R(X, Y, W, V) + (\phi - 1)P_6(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_6(\nabla\phi, \phi, X, Y, V, W). \\
I_7 &= \hat{R}(X, W, Y, V) = R(X, W, Y, V) + (\phi - 1)P_7(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_7(\nabla\phi, \phi, X, Y, V, W) + \frac{1}{2}Hess(\phi)(X, Y)g(V, W). \\
I_8 &= \hat{R}(V, W, W, X) = R(V, W, W, X) + (\phi - 1)P_8(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_8(\nabla\phi, \phi, X, Y, V, W) + \frac{1}{2}g(V, W)g(\nabla_W \nabla\phi, X) - \frac{1}{2}g(W, W)g(\nabla_V \nabla\phi, X). \\
I_9 &= \hat{R}(W, V, V, Y) = R(W, V, V, Y) + (\phi - 1)P_9(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q_9(\nabla\phi, \phi, X, Y, V, W) + \frac{1}{2}g(V, W)g(\nabla_V \nabla\phi, Y) - \frac{1}{2}g(V, V)g(\nabla_W \nabla\phi, Y),
\end{aligned}$$

where $P_i(\nabla\phi, \phi, X, Y, V, W)$, $Q_i(\nabla\phi, \phi, X, Y, V, W)$ are two functions depending on $\nabla\phi, \phi, X, Y, V, W$ and $Q_i(0, \phi, X, Y, V, W) \equiv 0$, $i = 3, 4, \dots, 9$. Hence

$$\begin{aligned}
\hat{R}(X + V, Y + W, Y + W, X + V) &= I_0 + I_1 + I_2 + I_3 + 2 \sum_{i=4}^9 I_i \\
&= R(X + V, Y + W, Y + W, X + V) + (\phi - 1)P(\nabla\phi, \phi, X, Y, V, W) \\
&\quad + Q(\nabla\phi, \phi, X, Y, V, W) + [-g(W, W)g(\nabla_V \nabla\phi, X) + g(V, W)g(\nabla_W \nabla\phi, X) \\
&\quad + g(V, W)g(\nabla_V \nabla\phi, Y) - g(V, V)g(\nabla_W \nabla\phi, Y)] \\
&\quad + \frac{1}{2}[-Hess(\phi)(X, X)g(W, W) + 2Hess(\phi)(X, Y)g(V, W) \\
&\quad - Hess(\phi)(Y, Y)g(V, V)],
\end{aligned}$$

where $P(\nabla\phi, \phi, X, Y, V, W)$, $Q(\nabla\phi, \phi, X, Y, V, W)$ are two functions which depend on $\nabla\phi, \phi, X, Y, V, W$ and $Q(0, \phi, X, Y, V, W) \equiv 0$. \square

Proof of Theorem 3.1:

Proof. We prove it by contradiction. We already proved it if $\|A\|$ is basic. Hence it suffices to show it if H is basic. We prove it by contradiction. Let $\pi : (M^m, g) \rightarrow (N^2, h)$ be a Riemannian submersion such that H is basic and the fibers have nonzero Euler numbers, where (M^m, g) has positive sectional curvature and $m \geq 4$. By Lemma 3.3, there exists a metric \hat{g} on M^m such that $\pi : (M^m, \hat{g}) \rightarrow (N^2, h)$ is still a Riemannian submersion and all fibers of π are minimal submanifolds with respect to \hat{g} . Furthermore, there exists some fiber F_0 such that \hat{g} has positive sectional curvature at all points in F_0 . Let r be a fixed positive number such that the normal exponential map of F_0 is a diffeomorphism when restricted to the tubular neighborhood of F_0 with radius r . By continuity of sectional curvature, there exists ϵ , $0 < \epsilon < r$ such that \hat{g} has positive sectional curvature at the ϵ neighborhood of F_0 . Choose another fiber F_1 such that $0 < \hat{d}(F_0, F_1) < \epsilon$, where $\hat{d}(F_0, F_1)$ is the distance between F_0 and F_1 with respect to \hat{g} . Since $\pi : (M^m, \hat{g}) \rightarrow (N^2, h)$ is a Riemannian submersion, F_0 and F_1 are equidistant. On the other hand, since $0 < \hat{d}(F_0, F_1) < \epsilon$, then for any point $q \in F_1$, there is a *unique* point $p \in F_0$ such that $\hat{d}(p, q) = \hat{d}(F_0, F_1)$. Let $L = \hat{d}(p, q)$ and $\gamma : [0, L] \rightarrow M^m, \gamma(0) = p, \gamma(L) = q$ be the *unique* minimal geodesic with unit speed realizing the distance between p and q . Let $V \subseteq T_q(M^m)$ be the subspace of vectors $v = X(L)$ where X is a parallel field along γ such that $X(0) \in T_p(F_0)$. Then

$$\begin{aligned} \dim(V \cap T_q(F_1)) &= \dim(V) + \dim(T_q(F_1)) - \dim(V + T_q(F_1)) \\ &\geq (m-2) + (m-2) - (m-1) = m-3. \end{aligned}$$

We claim that $\dim(V \cap T_q(F_1)) = m-3$. If not, then $\dim(V \cap T_q(F_1)) = m-2$. Let $X_i, i = 1, \dots, m-2$, be orthonormal parallel fields along γ such that $X_i(0) \in T_p(F_0), X_i(L) \in T_q(F_1)$. For each i , choose a variation $f_i(s, t)$ of γ such that $f_i(s, 0) \in F_0, f_i(s, L) \in F_1$ for small s and $\frac{\partial f_i(0, t)}{\partial s} = X_i(t)$. By construction, $\dot{X}_i(t) = \hat{\nabla}_{\dot{\gamma}} X_i(t) = 0$ for all t , where $\hat{\nabla}$ is the Levi-Civita connection with respect to \hat{g} . By the second variation formula, for $i = 1, \dots, m-2$, we have

$$\begin{aligned} \frac{1}{2} \frac{d^2 E_i(s)}{ds^2} \Big|_{s=0} &= \int_0^L (\hat{g}(\dot{X}_i, \dot{X}_i) - \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i)) dt \\ &\quad + \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0) \\ &= - \int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) dt + \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0), \end{aligned}$$

where $E_i(s) = \int_0^L \hat{g}(\frac{\partial f_i(s, t)}{\partial t}, \frac{\partial f_i(s, t)}{\partial t}) dt$, \hat{R} is the curvature tensor of \hat{g} and \hat{B}_j is the second fundamental form of F_j with respect to \hat{g} , $j = 0, 1$.

Since F_0 and F_1 are minimal submanifolds in (M^m, \hat{g}) , we have

$$\sum_{i=1}^{m-2} \hat{B}_j(X_i, X_i) = 0, j = 0, 1.$$

Then

$$\frac{1}{2} \sum_{i=1}^{m-2} \frac{d^2 E_i(s)}{ds^2} \Big|_{s=0} = - \sum_{i=1}^{m-2} \int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) dt.$$

Since \hat{g} has positive sectional curvature at the ϵ neighborhood of F_0 and $0 < \hat{d}(F_0, F_1) < \epsilon$, we see that $\hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) < 0$. Hence

$$\frac{1}{2} \sum_{i=1}^{m-2} \frac{d^2 E_i(s)}{ds^2} \Big|_{s=0} < 0.$$

Then there exists some i_0 such that $\frac{d^2 E_{i_0}(s)}{ds^2} \Big|_{s=0} < 0$, which contradicts that γ is a minimal geodesic realizing the distance between F_0 and F_1 . So $\dim(V \cap T_q(F_1)) = m - 3$. Since $\dim(T_q(F_1)) = m - 2$, then $V \cap T_q(F_1)$ is a codimension one subspace of $T_q(F_1)$. Since q is arbitrary on F_1 , by doing the same construction as above for any q , then we get a continuous codimension one distribution on F_1 . Thus the Euler number of F_1 is zero. Contradiction. \square

4 Proof of theorem 1.4

In this section we prove Theorem 1.4. Suppose $\pi : (M^4, g) \rightarrow (N^2, h)$ is a Riemannian submersion with totally geodesic fibers, where (M^4, g) is a compact four-dimensional Einstein manifold. We are going to show that the A tensor of π vanishes and then locally π is the projection of a metric product onto one of the factors. We firstly need the following lemmas:

Lemma 4.1. *Let π be a Riemannian submersion with totally geodesic fibers from compact Riemannian manifolds, then all fibers are isometric to each other.*

Proof. See [12]. \square

Lemma 4.2. *Suppose $\pi : (M^4, g) \rightarrow (N^2, h)$ is a Riemannian submersion with totally geodesic fibers, where (M^4, g) is a compact four-dimensional Einstein manifold. Let c_1, c_2 be the sectional curvature of $(F^2, g|_{F^2})$ and (N^2, h) , respectively, where $g|_{F^2}$ is the restriction of g to the fibers F^2 . Let $\text{Ric}(g) = \lambda g$ for some λ . Then*

$$(i) \quad 2c_1 + \|A\|^2 = 2\lambda;$$

$$(ii) \quad 2c_2 \circ \pi - 2\|A\|^2 = 2\lambda;$$

$$(iii) \quad \|A\|^2 = \frac{2}{3}(c_2 \circ \pi - c_1),$$

where $\|A\|^2 = \|A_X^* U\|^2 + \|A_X^* V\|^2 + \|A_Y^* U\|^2 + \|A_Y^* V\|^2$. Here $X, Y/U, V$ is an orthonormal basis of \mathcal{H}/\mathcal{V} , respectively.

Proof. See page 250, Corollary 9.62 in [3]. For completeness, we give a proof here.

Let $U, V / X, Y$ are orthonormal basis of $\mathcal{V} / \mathcal{H}$, respectively. Then by O'Neill's formula ([14]), we have

$$\lambda = \text{Ric}(U, U) = c_1 + \|A_X^* U\|^2 + \|A_Y^* U\|^2;$$

$$\lambda = \text{Ric}(V, V) = c_1 + \|A_X^* V\|^2 + \|A_Y^* V\|^2;$$

$$\lambda = \text{Ric}(X, X) = c_2 \circ \pi - 3\|A_X Y\|^2 + \|A_X^* U\|^2 + \|A_X^* V\|^2;$$

$$\lambda = \text{Ric}(Y, Y) = c_2 \circ \pi - 3\|A_X Y\|^2 + \|A_Y^* U\|^2 + \|A_Y^* V\|^2.$$

On the other hand, by direct calculation, we see that $2\|A_X Y\|^2 = \|A\|^2$. Hence

$$\begin{aligned} 2c_1 + \|A\|^2 &= 2\lambda; \\ 2c_2 \circ \pi - 2\|A\|^2 &= 2\lambda; \\ \|A\|^2 &= \frac{2}{3}(c_2 \circ \pi - c_1). \end{aligned}$$

□

By Lemmas 4.1 and 4.2, we see that $c_1, \|A\|$ are constants on M^4 and c_2 is a constant on N^2 .

Fix $p \in M^4$. Locally we can always choose basic vector fields X, Y such that X, Y is an orthonormal basis of the horizontal distribution. At point p , since the image of A_X^* is perpendicular to X and $\dim \mathcal{V} = \dim \mathcal{H} = 2$, A_X^* must have nontrivial kernel. Then there exists some $v \in \mathcal{V}$ such that $\|v\| = 1$ and $A_X^*(v) = 0$. Extend v to be a local unit vertical vector field V and choose U such that U, V is a local orthonormal basis of \mathcal{V} .

Lemma 4.3.

$$\begin{aligned} A_X^* V(p) &= 0; \\ A_Y^* V(p) &= 0. \end{aligned}$$

Proof. We already see $A_X^* V(p) = A_{X,p}^*(v) = 0$. On the other hand, at point p , we have

$$\begin{aligned} A_Y^* V &= g(A_Y^* V, X)X = -g(\nabla_Y V, X)X \\ &= g(V, \nabla_Y X)X = g(V, A_Y X)X \\ &= -g(V, A_X Y)X = -g(V, \nabla_X Y)X \\ &= g(\nabla_X V, Y)X = -g(A_X^* V, Y)X = 0. \end{aligned}$$

□

Since all fibers of π are totally geodesic, by O'Neill's formula ([14]), we see that $K(X, U) = \|A_X^* U\|^2$. Because (M^4, g) is Einstein, at point p , we have

$$\begin{aligned} \lambda &= Ric(U, U) = c_1 + \|A_X^* U\|^2 + \|A_Y^* U\|^2; \\ \lambda &= Ric(V, V) = c_1 + \|A_X^* V\|^2 + \|A_Y^* V\|^2; \end{aligned}$$

Combined with Lemma 4.3, we see that $\lambda = c_1$ and $\|A_X^* U\|^2(p) = 0$, $\|A_Y^* U\|^2(p) = 0$. Then $\|A\|^2(p) = 0$. Hence $\|A\|^2 \equiv 0$ on M^4 and $c_1 = c_2$. Let $c = c_1 = c_2$. Then locally π is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors, where $B^2(c)$ is a two-dimensional compact manifold with constant curvature c .

5 Conjecture 1 and the Weak Hopf Conjecture

In this section we point out several interesting corollaries of Conjecture 1.

Suppose (E, g) is a complete, open Riemannian manifold with nonnegative sectional curvature. By a well known theorem of Cheeger and Gromoll [4], E contains a compact totally geodesic submanifold Σ , called the soul, such that E is diffeomorphic to the normal bundle of Σ . Let Σ_r be the distance sphere to Σ of radius r . Then for small $r > 0$, the induced metric on Σ_r has nonnegative sectional curvature by a theorem of Guijarro and Walschap [10]. In [9], Gromoll and Tapp proposed the following conjecture:

Weak Hopf Conjecture *Let $k \geq 3$. Then for any complete metric with nonnegative sectional curvature on $S^n \times \mathbb{R}^k$, the induced metric on the boundary of a small metric tube about the soul can not have positive sectional curvature.*

The case $n = 2, k = 3$ is of particular interest since the metric tube of the soul is diffeomorphic to $S^2 \times S^2$.

Recall that a map between metric spaces $\sigma : X \rightarrow Y$ is a submetry if for all $x \in X$ and $r \in [0, r(x)]$ we have that $f(B(x, r)) = B(f(x), r)$, where $B(p, r)$ denotes the open metric ball centered at p of radius x and $r(x)$ is some positive continuous function. If both X and Y are Riemannian manifolds, then σ is a Riemannian submersion of class $C^{1,1}$ by a theorem of Berestovskii and Guijarro [1].

Proposition 5.1. *Suppose Σ is a soul of (E, g) , where (E, g) is a complete, open Riemannian manifold with nonnegative sectional curvature. If the induced metric on Σ_r has positive sectional curvature at some point for some $r > 0$, then there is a Riemannian submersion from Σ_r to Σ with fibers S^{l-1} , where $l = \dim(E) - \dim(\Sigma)$.*

Proof. In fact, by a theorem of Guijarro and Walschap in [11], if Σ_r has positive sectional curvature at some point, the normal holonomy group of Σ acts transitively on Σ_r . By Corollary 5 in [24], we get a submetry $\pi : (E, g) \rightarrow \Sigma \times [0, +\infty)$ with fibers S^{l-1} , where $\Sigma \times [0, +\infty)$ is endowed with the product metric. Then $\pi : (\pi^{-1}(\Sigma \times (0, +\infty)), g) \rightarrow \Sigma \times (0, +\infty)$ is also a submetry. By a theorem of Berestovskii and Guijarro in [1], π is a $C^{1,1}$ Riemannian submersion. Then $\Sigma_r = \pi^{-1}(\Sigma \times \{r\})$ and $\pi : \Sigma_r \rightarrow \Sigma$ is also a $C^{1,1}$ Riemannian submersion with fibers S^{l-1} , where Σ_r is endowed with the induced metric from (E, g) . \square

Proposition 5.2. *When $k > n$, Conjecture 1 implies Weak Hopf Conjecture.*

Proof. Suppose for some complete metric g on $S^n \times \mathbb{R}^k$ with nonnegative sectional curvature, the induced metric on Σ_r has positive sectional curvature for some $r > 0$, where Σ is a soul. Since $S^n \times \mathbb{R}^k$ is diffeomorphic to the normal bundle of Σ , we see that Σ is a homotopy sphere and $\dim(\Sigma) = n$. By Proposition 5.1, we get a Riemannian submersion from Σ_r to Σ with fibers S^{k-1} , where Σ_r is endowed with the induced metric from g and hence has positive sectional curvature. Since $k > n$, we see $k - 1 \geq n$, which is impossible if Conjecture 1 is true for $C^{1,1}$ Riemannian submersions. \square

Remark 3. *If Remark 1 in section 3 is true, then by Proposition 5.1 again, any small metric tube about the soul can not have positive sectional curvature when the soul is odd-dimensional. This would give a solution to a question asked by K. Tapp in [19].*

References

- [1] V. N. Berestovskii and L. Guijarro, A metric characterization of Riemannian submersions. *Ann. Global Anal. Geom.* 18 (2000), no. 6, 577-588.
- [2] M. Berger, Sur les variétés d'Einstein compactes, *C. R. IIP Réunion Math. Expression Latine*, Namur (1965) 35-55.
- [3] A. L. Besse, *Einstein Manifolds*. Reprint of the 1987 edition. *Classics in Mathematics*. Springer-Verlag, Berlin, 2008.
- [4] J. Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative curvature. *Ann. of Math.* 96 (1972), no. 3, 413-443.
- [5] X. Chen, *Curvature and Riemannian submersions*. Ph.D thesis, University of Notre Dame, 2014.
- [6] L. Florit and W. Ziller. Topological obstructions to fatness. *Geom. Topol.* 15 (2011), no. 2, 891-925.
- [7] T. Frankel, Manifolds with positive curvature. *Pacific J. Math.* 11 (1961), 165-174.
- [8] D. Gromoll and K. Grove, The low-dimensional metric foliations of Euclidean spheres. *J. Differential Geom.* 28 (1988), no. 1, 143-156.
- [9] D. Gromoll and K. Tapp, Nonnegatively curved metrics on $S^2 \times \mathbb{R}^2$. *Geom. Dedicata.* 99 (2003), 127-136.
- [10] L. Guijarro and G. Walschap, The metric projection onto the soul. *Trans. Amer. Math. Soc.* 352 (2000) no. 1, 55-69.
- [11] L. Guijarro and G. Walschap, The dual foliation in open manifolds with nonnegative sectional curvature. *Proc. Amer. Math. Soc.* 136 (2008) no. 4, 1419-1425.
- [12] R. Hermann, A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. *Proc. Amer. Math. Soc.* 11 (1960), 236-242.
- [13] N. Hitchin, Compact four-dimensional Einstein manifolds. *J. Differential Geom.* 9 (1974), 435-441.
- [14] B. O'Neill, The fundamental equations of a submersion. *Michigan Math. J.* 13 (1966), 459-469.
- [15] P. Petersen, *Riemannian geometry*. Second edition. *Graduate Texts in Mathematics*, 171. Springer, New York, 2006.
- [16] A. Petrunin, *Exercises in orthodox geometry*, arXiv:0906.0290v6 [math.HO], 2013.
- [17] V. Rovenski, Foliations, submanifolds, and mixed curvature. *J. Math. Sci. (New York)* 99 (2000), no. 6, 1699-1787.
- [18] K. Tapp, Metrics with nonnegative curvature on $S^2 \times \mathbb{R}^4$, *Ann. Global Anal. Geom.* 42 (2012), no. 1, 61-77.

- [19] K. Tapp, Rigidity for odd-dimensional souls. *Geom. Topol.* 16 (2012), no. 2, 957-962.
- [20] P. Tondeur, *Geometry of foliations*. Monographs in Mathematics, 90. Birkhäuser Verlag, Basel, 1997.
- [21] G. Walschap, Metric foliations and curvature. *J. Geom. Anal.* 2 (1992), no. 4, 373-381.
- [22] B. Wilking, Index parity of closed geodesics and rigidity of Hopf fibrations. *Invent. Math.* 144 (2001), no. 2, 281-295.
- [23] B. Wilking, Manifolds with positive sectional curvature almost everywhere. *Invent Math.* 148 (2002), no. 1, 117-141.
- [24] B. Wilking, A duality theorem for Riemannian foliations in nonnegative sectional curvature, *Geom. Funct. Anal.* 17 (2007), no. 4, 1297-1320.

Department of Mathematics
 University of Notre Dame
 Notre Dame, Indiana, 46556.
 E-mail address: *xychen100@gmail.com*